MATH 1650: SECTION 5.1: GRAPHS OF FUNCTIONS

DEFINITION: A function is called:

- **even** if and only if $f(-x) = f(x)$ for all x in the domain of f.
  In this case, the graph of $y = f(x)$ is symmetric about the y-axis.
- **odd** if and only if $f(-x) = -f(x)$ for all x in the domain of f.
  In this case, the graph of $y = f(x)$ is symmetric about the origin.

EXAMPLE: Determine whether or not the following functions are even, odd, or neither even nor odd. Check your answer using a graphing utility.

- $p(x) = 3x^4 - x^2 + 1$

  We find $p(-x) = 3(-x)^4 - (-x)^2 + 1 = 3x^4 - x^2 + 1$ which matches our given formula for $p(x)$.

  Hence, p is even. Graphing $y = p(x)$ suggests the graph is, indeed, symmetric about the y-axis.

- $f(x) = \dfrac{x^3 - 4x}{x^2 + 1}$.

  We find $f(-x) = \dfrac{(-x)^3 - 4(-x)}{(-x)^2 + 1} = \dfrac{-x^3 + 4x}{x^2 + 1}$.

  Since $f(-x)$ doesn't seem to match $f(x)$, we **suspect** f is not even.

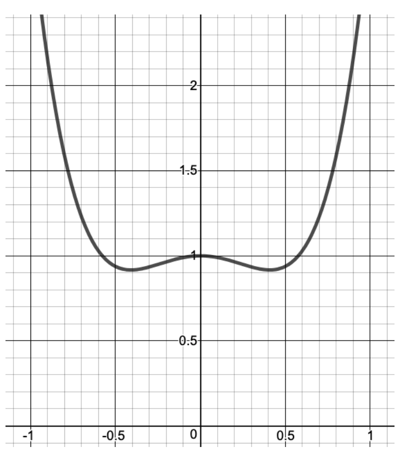
  To **prove** f is not even, we find $f(1) = \dfrac{(1)^3 - 4(1)}{(1)^2 + 1} = -\dfrac{3}{2}$ and $f(-1) = \dfrac{(-1)^3 - 4(-1)}{(-1)^2 + 1} = \dfrac{3}{2}$.

  Since $f(-1) \neq f(1)$, we have proven that f is not even.

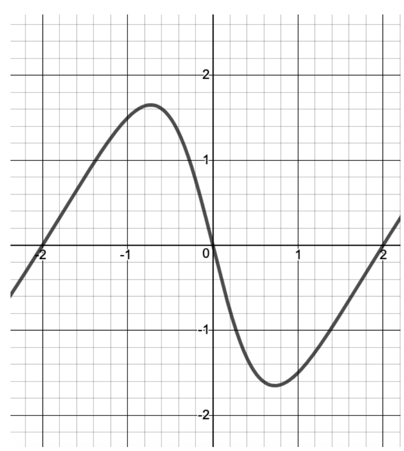
  Next, we try to see if f is odd. We find $-f(x) = -\dfrac{x^3 - 4x}{x^2 + 1} = \dfrac{-\left(x^3 - 4x\right)}{x^2 + 1} = \dfrac{-x^3 + 4x}{x^2 + 1}$.

  Since $f(-x) = -f(x)$, we have f is odd.

  Geometrically, the graph of $y = f(x)$ appears to be symmetric about the origin.



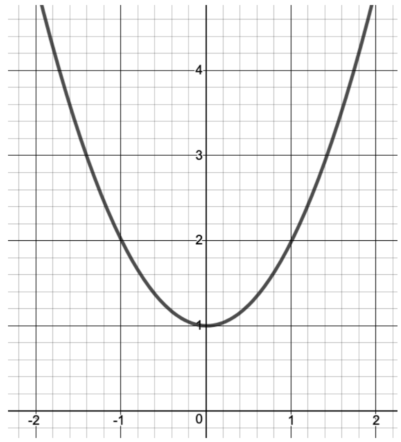
$y = p(x) = 3x^4 - x^2 + 1$



$y = f(x) = \dfrac{x^3 - 4x}{x^2 + 1}$.

EXAMPLE: Let $f(x) = x^2 - 0.02x + 1$.

- Graph $y = f(x)$ using a graphing utility, Does f appear to be even, odd, or neither even nor odd?



The graph of $y = f(x)$ certainly appears to be symmetric about the y-axis, so this suggests f is even.

- Prove f is, in fact, neither even nor odd.

We find $f(1) = (1)^2 - 0.02(1) + 1 = 1.98$ and $f(-1) = (-1)^2 - 0.02(-1) + 1 = 2.02$.

Since $f(-1) \neq f(1)$, f is not even. Since $f(-1) \neq -f(1)$, f is not odd.

Hence, despite appearances, f is neither even nor odd.

EXAMPLE: Determine whether the following functions are even, odd, or neither even nor odd. Check your answer graphically.

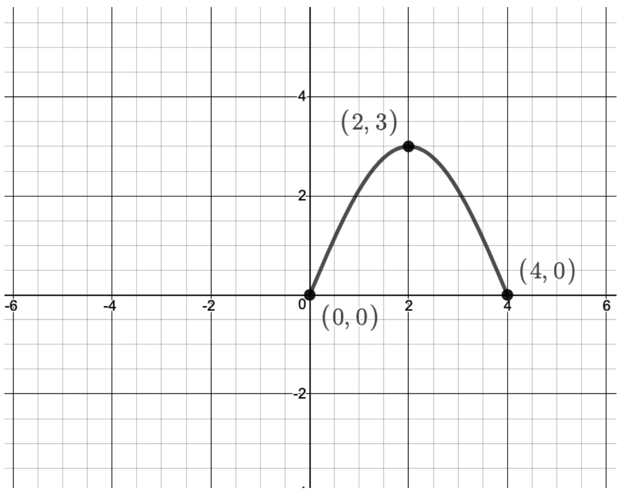
- $f(x) = 3x\sqrt{x^2 - 1}$

- $g(x) = \dfrac{x^2 + 1}{x^2 - 1}$

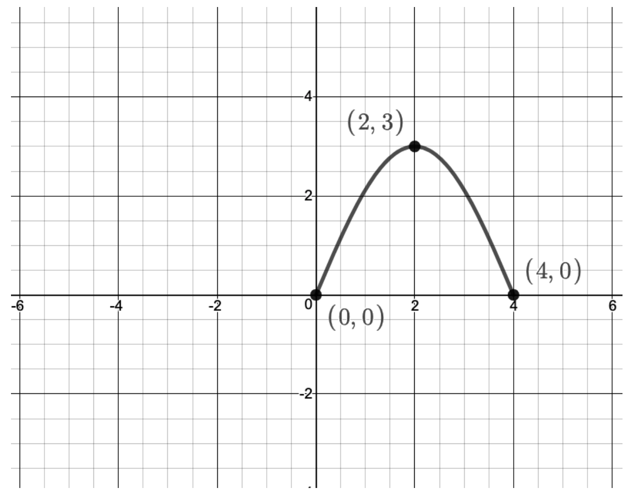
- $h(x) = 3x^2 - 2x + 1$.

EXAMPLE: Below is the (partial) graph of a function, f. Complete each graph as directed:

Complete the graph assuming f is **even**.



Complete the graph assuming f is **odd**.



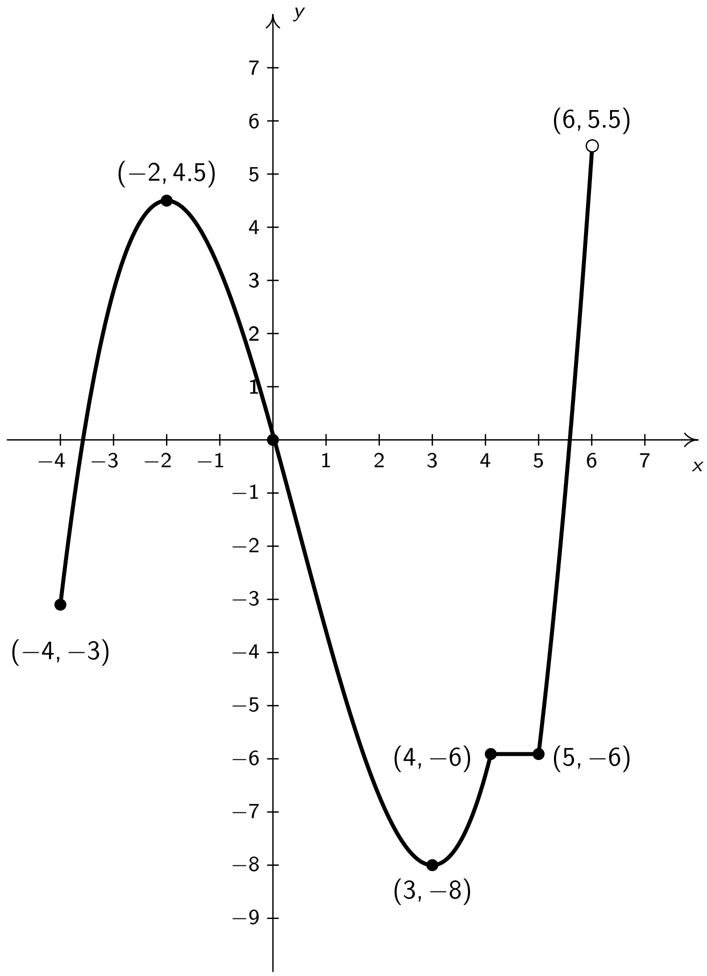
GENERAL CHARACTERISTICS OF GRAPHS OF FUNCTIONS:

- Suppose f is a function defined on an interval I. We say f is:
  - **increasing** on I if and only if for all real numbers a, b in I with $a < b$, $f(a) < f(b)$.
    That is, as your inputs increase, your outputs increase; as you move from left to right, the graph rises.
  - **decreasing** on I if and only if for all real numbers a, b in I with $a < b$, $f(a) > f(b)$.
    That is, as your inputs increase, your outputs decrease; as you move from left to right, the graph falls.
  - **constant** on I if and only if for all real numbers a, b in I, $f(a) = f(b)$.
    That is, as your inputs increase, your outputs don't change; the graph is horizontal.

- Suppose f is a function with $f(a) = b$.
  - We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f.'
    i.e., there are no points on the graph 'near' (a, b) which are higher on the graph than (a, b).
  - We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f.'
    i.e., there are no points on the graph 'near' (a, b) which are lower on the graph than (a, b).
  - The value b is called the **maximum** of f if $b \geq f(x)$ for all x in the domain of f.
  - The value b is called the **minimum** of f if $b \leq f(x)$ for all x in the domain of f.

  NOTE: In the definition of 'local' maximum and minimum, the requirement of an 'open' interval means there needs to be graph on both sides of the point with which to compare.

EXAMPLE: Answer the following questions using the graph of f:



- State the domain.
- State the range.
- Find $f(-2)$
- Find the y-intercept.
- Approximate the zeros of f.
- Solve $f(x) = -6$
- State where f is increasing.
- State where f is decreasing.
- State where f is constant.
- List the local maximums.
- List the local minimums.
- List the minimum.

Does f have a maximum? Explain.

HOMEWORK: Section 5.1: 1 - 41 every other odd.